

# On-Shell Description of Unsteady Flames

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The problem of non-perturbative description of unsteady premixed flames with arbitrary gas expansion is solved in the two-dimensional case. Considering the flame as a surface of discontinuity with arbitrary local burning rate and gas velocity jumps given on it, we show that the front dynamics can be determined without having to solve the flow equations in the bulk. On the basis of the Thomson circulation theorem, an implicit integral representation of the gas velocity downstream is constructed. It is then simplified by a successive stripping of the potential contributions to obtain an explicit expression for the vortex component near the flame front. We prove that the unknown potential component is left bounded and divergence-free by this procedure, and hence can be eliminated using the dispersion relation for its on-shell value (i.e., the value along the flame front). The resulting system of integro-differential equations relates the on-shell fuel velocity and the front position. As limiting cases, these equations contain all theoretical results on flame dynamics established so far, including the linear equation describing the Darrieus-Landau instability of planar flames, and the nonlinear Sivashinsky-Clavin equation for flames with weak gas expansion.

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## I. INTRODUCTION

Although the elementary physical mechanisms underlying flame propagation are well-understood by now, a global mathematical description of the process is extremely difficult. The reasons – which flames share with condensation discontinuities in supersaturated vapors [1], ablation fronts driven by lasers or X-rays [2], thermonuclear waves in type-Ia supernovæ [3] or rapid decomposition of explosive liquids [1] – can be summarized briefly: all involve propagating free-boundaries crossed by nonzero mass fluxes and separating subsonic flows of markedly different densities. In view of this it is not surprising that, since their identification as self-propagating slow deflagrations (as opposed to detonations) in the 1880s [4], flames had to wait some fifty years before their theoretical study began at the simplest level of linear stability analysis of planar flames [5, 6]. It took some forty years more for the first consistent account of nonlinear effects to appear. It was shown by [7] how these effects can be described in the case when the fresh to burnt gas density ratio,  $\theta$ , is close to unity. The latter condition is a principle limitation for a *perturbative* treatment of nonlinear saturation phenomena. Indeed, the weak nonlinearity expansions are simply self-contradictory in the case of steady flames with a finite  $(\theta - 1)$  [8]. Despite the fact that the practically relevant values of  $\theta$  are 5 to 8, the small  $(\theta - 1)$  approximation has so far been the sole theoretical method<sup>1</sup> available to handle the flame front dynamics.

One of the essential difficulties encountered in any analytic treatment of flames is the virtual impossibility to solve the flow equations governing the dynamics of the burned gases. This is an exceedingly complicated problem which requires finding solutions to a system of nonlinear partial differential equations in the regions ahead of the flame front and behind it, to be chosen so as to satisfy a number of jump conditions expressing the conservation of mass, energy and momentum across the moving front. The front dynamics itself is determined by the so-called evolution equation describing the local fuel consumption rate as a functional of the fuel velocity distribution along the front and the front shape [10]. Even if the gas flow is potential upstream, as is the case for flames propagating in an initially quiescent fluid, this property is lost in the downstream region because vorticity is generated by the curved flame front, so that the problem of solving the flow equations is faced in its full generality.

Struggling with this problem is indeed unavoidable if one is interested in the explicit structure of the burnt gas flow. However, it is the evolution of the flame front, its position and shape that usually constitute the main concern in practice. This limitation of the problem raises naturally the following *dilemma* [see 11, 12]: On the one hand,

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<sup>1</sup> We do not consider models inconsistent from the standpoint of the fundamental equations, such as for instance the Frankel's [9] potential-flow model. The latter assumes finite values of  $(\theta - 1)$  and zero vorticity production in the flame front at the same time; these two assumptions are only consistent in the lowest order of the small  $(\theta - 1)$  expansion, in which case the Frankel equation reduces to the Sivashinsky equation [7].

deflagration is an essentially non-local and nonlinear process with all its complications mentioned above; on the other hand, this non-locality itself is determined by the flame front configuration and the gas velocity distribution along it, which play the role of boundary conditions for the flow equations and thus control the bulk flow. In such circumstances, is it really necessary to know explicitly the flow structure in the entire downstream region in order to describe the front evolution in a closed form, i.e. in a form involving only quantities defined on the flame front?

As to the steady configurations, this question was settled out in the negative by [11, 12]. More precisely, it was shown that the only piece of information about the gas flow downstream that is really necessary to derive an equation for the flame front position is the value of the vortex component of burnt gas along the front or, using the terminology used by [12], the *on-shell value* of this component. The remaining unknown potential component of the gas velocity is excluded eventually using a “dispersion relation” for its on-shell values (expressing analyticity of this component in the downstream region), thus providing one with a description of flame shapes in a form which is closed in the above-mentioned sense. The purpose of the present paper is to generalize this construction to the case of unsteady flame propagation. Although the procedure is essentially the same as in the steady case, a subtle point is worth to be emphasized. According to [11, 12], the derivative of the vortex component along the flame front is local, i.e., its value at a given point is a function of the on-shell fuel velocity, its derivatives, and the front shape at the same point. In view of this, one might expect the generalization to the unsteady case to be purely kinematic, namely, that it would amount to rewriting the steady equation in terms of the gas velocities relative to the local front velocity. We will see, however, that this is not so because of a peculiar – yet unavoidable – spatial non-locality of the vortex component, which appears naturally in the unsteady case to account for the effect of the flame history on its current evolution. Because of this complication, one has to be more careful with the spatial integrations involved in the definition of the vorticity mode. To define the improper integrals along the flame front, we use an intermediate regularization. Specifically, we introduce an exponential damping of the contributions coming from remote parts of the front. This regularization is eventually removed using analytic continuation to the case of zero damping.

The paper is organized as follows. We first construct an implicit integral of the flow equations downstream on the basis of the Thomson circulation theorem, which expresses the gas velocity in terms of its boundary values and vorticity distribution behind the front. This is done in Sec. II. The integral representation is then used in Sec. III to obtain an expression for the vortex mode of the gas velocity near the front, which is accomplished by successive stripping burnt gas velocity of potential contributions. We prove that the unknown potential component is left bounded and divergence-free by this procedure. Hence, it can be eliminated using the dispersion relation for its on-shell value, thus leading to the main integro-differential equation written down in Sec. IV. This equation relates the on-shell value of the fuel velocity and the flame front position, and together with the evolution equation constitutes the closed system for these quantities. Finally, it is verified in Sec. V that the derived equation contains as simple limiting cases all known theoretical results on flame dynamics, namely, the linear equation describing Darrieus-Landau instability of planar flames [5, 6], including the case when the flame propagates in a time-dependent gravitational field [13], the nonlinear Sivashinsky-Clavin equation for flames with weak gas expansion [7, 14], and the stationary equation derived by [12]. Appendix contains a consistency check for the results obtained in Sec. III.

## II. INTEGRAL REPRESENTATION OF THE FLOW EQUATIONS

Consider a flame propagating in an initially uniform premixed ideal fluid. Our analysis below relies substantially on the well-known Thomson theorem stating that circulation of the gas velocity over any closed material contour drawn in an ideal fluid is conserved as it is convected. This statement takes an especially simple form in the case of two-dimensional (2D) incompressible flows, since not only the circulation itself, but also the value of vorticity carried by any fluid element is then conserved. Since the space dimensionality is not that important in the *formulation* (not the resolution) of the dilemma mentioned in the Introduction, we will be concerned in what follows with the simpler 2D case. We will further specify our analysis to flames propagating in a channel of constant width  $b$ . Let the Cartesian coordinates  $(x, y)$  be chosen so that the  $y$ -axis is parallel to the tube walls,  $y = -\infty$  being in the fresh fuel. These coordinates will be measured in units of the channel width, while the fluid velocity,  $\mathbf{v} = (v_1, v_2)$ , in units of the velocity of a plane flame front relative to the fuel. It will be sometimes useful to denote the Cartesian components of  $\mathbf{v}$  by  $(w, u)$ . Finally, the fluid density will be normalized on the fuel density,  $\theta > 1$  denoting its ratio to that of the burnt gas.

It will be more appropriate for our purposes to reformulate the problem under consideration as a problem of propagation of an unbounded spatially-periodic flame. Namely, assuming the channel walls ideal, given a flame configuration described by the functions  $f(x, t), \mathbf{v}(x, y, t)$ ,  $x \in [0, +1]$ , where  $f(x, t)$  denotes the flame front position at time instant  $t$ , using the boundary conditions  $f' = 0, w = 0$  for  $x = 0, 1$ , we continue this configuration to the domain  $x \in [-1, 0]$  according to

$$f(x, t) = f(-x, t), \quad w(x, y, t) = -w(-x, y, t), \quad u(x, y, t) = u(-x, y, t), \quad (1)$$

and then periodically continue it to the whole  $x$ -axis. Note that having imposed the boundary condition  $f'(0) = f'(1) = 0$ , we thereby exclude the possibility of stagnation zone formation near the end points of the flame front (see [15] for detail). We also assume that the flame is stable with respect to the short wavelength perturbations i.e., that there is a short wavelength cutoff,  $\lambda_c$ . This cutoff ensures smoothness of the functions under consideration. In particular, it prevents the development of singularities of the front shape such as the edge points which would occur otherwise [15], leading to discontinuities in the values of the flow variables or their derivatives. That  $\lambda_c$  often exceeds the actual thickness of the flame preheat zone significantly [16] has yet another virtue: the Reynolds number based on  $\lambda_c$  and the fuel properties is typically over  $\sim 10^2$ , and hence is fairly large when based upon the width ( $> \lambda_c$  or  $\gg \lambda_c$ ) of the channel where the flame studied below is meant to propagate. It then makes sense to model the flame as a surface (or line in 2-D) equipped with a local  $\lambda_c$ -dependent propagation law, and embedded in ideal fluid flows. We shall return to this issue in the final section of the paper, merely mentioning here that viscosity effects are known from direct numerical simulations [17] to have negligible influence on the shape and the speed of steady curved flames.

In our formulation, the flow velocity obeys the following equations in the bulk

$$\frac{\partial v_i}{\partial x_i} = 0, \quad (2)$$

$$\frac{\partial \sigma}{\partial t} + v_i \frac{\partial \sigma}{\partial x_i} = 0, \quad (3)$$

where  $(x_1, x_2) = (x, y)$ ,

$$\sigma = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \quad (4)$$

is the vorticity, and summation over repeated indices is understood. The first equation expresses continuity of the incompressible flows, while the other the Thomson theorem. It is not difficult to see that  $\mathbf{v}$  satisfying Eqs. (2), (4) can be written in the following integral form (Cf. derivation of Eq. (9) in Kazakov 12)

$$v_i = \varepsilon_{ik} \partial_k \int_{\Lambda} dl_l \varepsilon_{lm} v_m \frac{\ln r}{2\pi} - \partial_i \int_{\Lambda} dl_k v_k \frac{\ln r}{2\pi} - \varepsilon_{ik} \partial_k \int_{\Sigma} ds \frac{\ln r}{2\pi} \sigma, \quad (5)$$

where  $\varepsilon_{ik} = -\varepsilon_{ki}$ ,  $\varepsilon_{12} = +1$ ,  $\partial_i = \partial/\partial x_i$ .  $\Sigma$  and  $\Lambda$  denote any part of the downstream region and its boundary, respectively.  $r$  is the distance between an infinitesimal fluid element  $ds$  at the point  $(\tilde{x}, \tilde{y})$  and the point of observation  $\mathbf{x} = (x, y) \in \Sigma$ ,  $r^2 = (x_i - \tilde{x}_i)^2$ , and  $dl_i$  is the line element normal to  $\Lambda$  and directed outwards of  $\Sigma$ . Indeed, to evaluate divergence of the right hand side of Eq. (5), we use the relations  $\partial_i \varepsilon_{ik} \partial_k \equiv 0$ ,

$$\partial_k^2 \ln r = 2\pi \delta(x - \tilde{x}) \delta(y - \tilde{y}), \quad (6)$$

where  $\delta(x)$  is the Dirac delta-function. It follows that  $\partial_i v_i = 0$  for any point  $\mathbf{x}$  inside  $\Sigma$ . Similarly, evaluating the curl of Eq. (5) with the help of Eq. (6) and  $\varepsilon_{ik} \varepsilon_{im} = \delta_{km}$  gives the identity  $\varepsilon_{ik} \partial_i v_k = \sigma$ . We are now to employ Eq. (3) to rewrite the last term in Eq. (5) as an integral over fluid particle trajectories.

To this end, we specify that  $\Sigma$  is spanned by fluid elements that crossed the flame front between two points on it with fixed abscissas  $\tilde{x}_1 = -A$  and  $\tilde{x}_2 = +A$  during the time interval  $[-T, t]$ , where  $t$  is the given time instant the flow is observed at (see Fig. 1). Here  $A, T$  are some positive parameters tending eventually to infinity, so that  $\Sigma$  will then fill the whole downstream region:  $\Sigma = \{\tilde{x}, \tilde{y} : \tilde{y} > f(\tilde{x}, t)\}$ . But for the time being, we keep them finite. The improper integral over an infinite  $\Sigma$  will be defined later, in Sec. III A. Then we have, by virtue of the vorticity conservation,

$$\begin{aligned} \int_{\Sigma} ds \sigma \ln r = \\ \int_{-A}^{+A} d\tilde{x} \int_{-T}^t d\tau N(\tilde{x}, \tau) \bar{v}_+^n(\tilde{x}, \tau) \sigma_+(\tilde{x}, \tau) \ln \left\{ [x - X(\tilde{x}, t, \tau)]^2 + [y - Y(\tilde{x}, t, \tau)]^2 \right\}^{1/2}, \end{aligned} \quad (7)$$

where  $N = \sqrt{1 + (\partial f / \partial x)^2}$ ,  $\bar{v}_+^n = \bar{v}_{i+} n_i$  is the normal burnt gas velocity relative to the flame front,

$$\bar{\mathbf{v}}_+ = (w_+, \bar{u}_+), \quad \bar{u}_+(x, t) \equiv u_+(x, t) - \frac{\partial f(x, t)}{\partial t},$$

$n_i$  is the unit vector normal to the front (pointing to the burnt matter), and  $(X(\tilde{x}, t, \tau), Y(\tilde{x}, t, \tau))$  is the current position of a fluid element that crossed the point  $(\tilde{x}, f(\tilde{x}, \tau))$  on the flame front at  $\tau$ . It is taken into account in Eq. (7) that the “volume”  $ds$  of this element is conserved in view of the flow incompressibility, and hence can be written as  $d\tilde{x}d\tau N(\tilde{x}, \tau)\bar{v}_+^n(\tilde{x}, \tau)$ . Changing the integration variable,  $\tau \rightarrow t - \tau$ , in the expression (7), and substituting it into Eq. (5) gives

$$v_i = \varepsilon_{ik}\partial_k \int_{\Lambda} dl_l \varepsilon_{lm} v_m \frac{\ln r}{2\pi} - \partial_i \int_{\Lambda} dl_k v_k \frac{\ln r}{2\pi} - \frac{\varepsilon_{ik}}{2} \partial_k \int_{-A}^{+A} d\tilde{x} K(x, y, \tilde{x}, t), \quad (8)$$

where the integral kernel  $K$  is defined by

$$K(x, y, \tilde{x}, t) = \frac{1}{\pi} \int_0^{T+t} d\tau M(\tilde{x}, t - \tau) \times \ln \{ [x - X(\tilde{x}, t, t - \tau)]^2 + [y - Y(\tilde{x}, t, t - \tau)]^2 \}^{1/2}, \quad (9)$$

$$M(\tilde{x}, \tau) \equiv N(\tilde{x}, t)\bar{v}_+^n(\tilde{x}, t)\sigma_+(\tilde{x}, t). \quad (10)$$

### III. NEAR-THE-FRONT STRUCTURE OF THE VORTEX MODE

The integral representation (8) of the flow velocity downstream is to be used below to obtain an expression for its vortex component near the flame front. More precisely, we are going to define a vortex component  $v_i^v$  in a way that would allow an explicit expression for its on-shell value in terms of the on-shell gas velocity  $v_{i+}$ . For this purpose, we decompose the velocity field as

$$v_i = v_i^p + v_i^v, \quad i = 1, 2,$$

where  $v_i^p$  is a potential component satisfying the following requirements ( $D\mathbf{v}^p$  denotes any first-order spatial derivative of  $\mathbf{v}^p$ ):

- a)  $\text{div} D\mathbf{v}^p = 0$ ,
- b)  $\text{rot} D\mathbf{v}^p = 0$ ,
- c)  $D\mathbf{v}^p$  is bounded, in the sense that it remains finite in the limit  $A, T \rightarrow +\infty$ , i.e., for an infinitely expanding  $\Sigma$  region.

The on-shell expression for the vorticity contribution will be obtained by a step-by-step simplification of Eq. (8) throwing away potential fields fulfilling the above conditions a)–c). Although this derivation follows closely that of [12], we give it here in detail to make clear the point where non-stationarity of the problem comes into play.

Let the equality of two functions  $\varphi_1(x, y)$ ,  $\varphi_2(x, y)$  up to a field satisfying a) – c) be denoted by  $\varphi_1 \overset{\circ}{=} \varphi_2$ . First of all, as we saw in the preceding section, the first two terms on the right hand side of Eq. (8) have vanishing curl and divergence, and hence satisfy also the requirements a) and b). Furthermore, a simple power counting shows that c) is also met. Indeed, consider part  $\Lambda \setminus F$  of the contour  $\Lambda$ , where  $F$  denotes the flame front. Representing this part as a semicircle with radius  $R \rightarrow \infty$ , we note that  $D^2 \ln r = O(1/R^2)$  for any given  $\mathbf{x} \in \Sigma$  ( $D^2$  denotes any second spatial derivative). Taking into account also that  $\mathbf{v} = O(1)$ ,  $dl = Rd\phi$ , where  $\phi \in (0, \pi)$  is the angular coordinate of the point  $\tilde{\mathbf{x}}$  on the semicircle, one sees that after spatial differentiation, the two integrals over  $\Lambda \setminus F$  on the right of Eq. (8) vanish in the limit  $R \rightarrow \infty$ . Similar consideration shows that the same integrals over  $F$  are convergent, thus proving that  $D\mathbf{v}^p$  remains bounded downstream in the limit  $A, T \rightarrow \infty$ . So we can write

$$v_i \overset{\circ}{=} -\frac{\varepsilon_{ik}}{2} \partial_k \int_{-A}^{+A} d\tilde{x} K(x, y, \tilde{x}, t). \quad (11)$$

Next, we note that since we are eventually interested in the on-shell value of the vortex component, we may take the observation point  $(x, y)$  as close to the flame front as we like, i.e.,  $y \approx f(x, t)$ ,  $[y > f(x, t)]$ . The vortex component at

such points is determined by a contribution coming from the integration over small  $\tau$  and  $\tilde{x} \approx x$ . Indeed, taking the curl of Eq. (11), and using Eq. (6) yields

$$\varepsilon_{ik} \partial_i v_k = \int_{-A}^{+A} d\tilde{x} \int_0^{T+t} d\tau M(\tilde{x}, t - \tau) \delta(x - X[\tilde{x}, t, t - \tau]) \delta(y - Y[\tilde{x}, t, t - \tau]),$$

which explicitly shows that a nonzero contribution to the vorticity comes only from the point  $\{\tilde{x}, \tau\}$  obeying the equations

$$x - X[\tilde{x}, t, t - \tau] = 0, \quad y - Y[\tilde{x}, t, t - \tau] = 0, \quad (12)$$

which for  $y \approx f(x, t)$  state that the point  $(X, Y)$  is close to the flame front, and hence

$$\begin{aligned} X(\tilde{x}, t, t - \tau) &= \tilde{x} + w_+(\tilde{x}, t)\tau + O(\tau^2), \\ Y(\tilde{x}, t, t - \tau) &= f(\tilde{x}, t - \tau) + u_+(\tilde{x}, t)\tau + O(\tau^2). \end{aligned}$$

Expanding also  $f(\tilde{x}, t - \tau)$  to the first order in  $\tau$  and omitting the symbols  $O(\tau^2)$ , we thus have the following approximate expression for the fluid particle trajectory

$$X(\tilde{x}, t, t - \tau) \approx \tilde{x} + w_+(\tilde{x}, t)\tau, \quad Y(\tilde{x}, t, t - \tau) \approx f(\tilde{x}, t) + \bar{u}_+(\tilde{x}, t)\tau. \quad (13)$$

It follows that if these expressions are used instead of the exact ones to calculate the kernel  $K$ , we still have the true distribution of vorticity near the flame front. Indeed, it was just shown that any integration over values of  $\{\tilde{x}, \tau\}$  not satisfying Eq. (12), where Eq. (13) is not valid either, gives rise to a potential contribution. In particular, the property b) of this contribution is preserved, and so is the property a), since the right hand side of Eq. (11) is divergence-free identically whatever the form of the kernel. Finally, it is not difficult to see that the condition c) is also satisfied. Indeed, the above transformation of exact trajectories into the straightened ones given by Eq. (13) leaves the expression (11) bounded. Therefore, the potential field added in the course of this transformation is bounded, too. It should be stressed that from now on we are concerned only with the on-shell value of the vortex component, so the meaning of the symbol  $\overset{\circ}{=}$  should be further specified. Of course, the transformation of trajectories changes the bulk vorticity distribution, and thereby the velocity field downstream. However, after the transformation as well as before it, integration over finite  $\tau$ s in Eq. (9) results in a field that is potential near the flame front. Both fields are of complicated structure which is unknown in general, but since they are potential near the front and bounded, we can use the on-shell value of their difference to define a field satisfying a)–c) in the whole downstream region. The existence of this field is guaranteed by the Cauchy theorem. Namely, we use the Cauchy formula to construct the field satisfying a)–c) as the analytic function with the given boundary value. It is in this sense that the above transformation is said to respect the properties a)–c). In particular, the symbol  $\overset{\circ}{=}$  is used below to relate the on-shell values (or near-the-front values, in the case of the integral kernel) of functions that differ by a field satisfying a)–c) downstream.

We now proceed to an explicit evaluation of the integral kernel (9) which, after the transformations (13) are performed, takes the form<sup>2</sup>

$$K(x, y, \tilde{x}, t) \overset{\circ}{=} \frac{1}{\pi} \int_0^{T+t} d\tau M(\tilde{x}, t - \tau) \ln \left\{ \bar{v}_+^2 \tau^2 - 2(\mathbf{r} \cdot \bar{\mathbf{v}}_+) \tau + r^2 \right\}^{1/2}, \quad (14)$$

where  $\mathbf{r} = (x - \tilde{x}, y - f(\tilde{x}, t))$ . The integrand here involves  $M(\tilde{x}, t - \tau)$  which is an unknown function of  $\tau$ . In view of what has been said about the near-the-front structure of the vorticity mode, one might think that it would be sufficient to expand this function to the first order in  $\tau$ , and then calculate the integral. However, this operation is not allowed as it would violate the condition c) and, as a result, would yield erroneous predictions (see Sec. 5). In particular, the on-shell value of the vorticity component cannot be found by setting  $\tau = 0$  in the argument of  $M$ . There is no such problem in the case of steadily propagating flames, as  $M$  is then time-independent in a frame of reference attached

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<sup>2</sup> When transforming the kernel  $K(x, y, \tilde{x}, t)$ , we use the same symbol  $\overset{\circ}{=}$  to relate expressions that upon substitution in Eq. (11) give rise to fields that are equal in the sense of  $\overset{\circ}{=}$ .

to the front. To overcome this difficulty, we will explicitly extract the singular part of the  $\tau$ -integral, which is related to the singularities of the logarithm located at the points

$$\tau_{\pm} = \frac{r}{\bar{v}_+} \left( \Omega \pm i\sqrt{1 - \Omega^2} \right), \quad \Omega = \frac{(\mathbf{r} \cdot \bar{\mathbf{v}}_+)}{r\bar{v}_+} \quad (15)$$

in the complex  $\tau$ -plane. For this purpose, we first formally integrate Eq. (14) by parts

$$\begin{aligned} K(x, y, \tilde{x}, t) &\stackrel{\circ}{=} -\frac{1}{\pi} \int_0^{T+t} d \left( \int_{\tau}^{T+t} d\tau_1 M(\tilde{x}, t - \tau_1) \right) \ln \{ \bar{v}_+^2 \tau^2 - 2(\mathbf{r} \cdot \bar{\mathbf{v}}_+) \tau + r^2 \}^{1/2} \\ &= \frac{\ln r}{\pi} \int_0^{T+t} d\tau_1 M(\tilde{x}, t - \tau_1) \\ &\quad + \frac{1}{2\pi} \int_0^{T+t} d\tau \left( \int_{\tau}^{T+t} d\tau_1 M(\tilde{x}, t - \tau_1) \right) \left\{ \frac{1}{\tau - \tau_+} + \frac{1}{\tau - \tau_-} \right\}. \end{aligned}$$

The first term on the right gives rise to a pure potential which is bounded, so it can be omitted. Denoting also

$$\int_{\tau}^{T+t} d\tau_1 M(\tilde{x}, t - \tau_1) \equiv \mathcal{M}(\tilde{x}, \tau, t), \quad (16)$$

we thus have

$$K(x, y, \tilde{x}, t) \stackrel{\circ}{=} \frac{1}{2\pi} \int_0^{T+t} d\tau \mathcal{M}(\tilde{x}, \tau, t) \left\{ \frac{1}{\tau - \tau_+} + \frac{1}{\tau - \tau_-} \right\}.$$

To extract the singular part of this integral, we deform the contour of integration in the complex  $\tau$ -plane so as to move it away from the poles; this of course implies certain time-wise restrictions on the function  $\mathcal{M}(\tilde{x}, \tau, t)$ , and hence on  $M(\tilde{x}, t)$ . We shall return to this matter later (see Sec. 7 and Appendix). Here we mention only that in essence, the function  $M(\tilde{x}, t)$  is required to be analytic in a vicinity of the real axis in the complex  $t$ -plane, which is guaranteed by the existence of nonzero  $\lambda_c$ . To respect reality of the kernel, we take the singular part as a half sum of two expressions obtained respectively by deforming the contour above and below the real axis. The contribution we are interested in comes from moving the contours beyond the poles (15) [see Fig. 2].

According to the Cauchy theorem,

$$\begin{aligned} K(x, y, \tilde{x}, t) &\stackrel{\circ}{=} \frac{2\pi i}{4\pi} \{ \mathcal{M}(\tilde{x}, \tau_+, t) - \mathcal{M}(\tilde{x}, \tau_-, t) \} \\ &\quad + \frac{1}{4\pi} \int_{C_- \cup C_+} d\tau \mathcal{M}(\tilde{x}, \tau, t) \left\{ \frac{1}{\tau - \tau_+} + \frac{1}{\tau - \tau_-} \right\}. \end{aligned} \quad (17)$$

Instead of proving that the integral over the contour  $C_- \cup C_+$  is free of singularity and gives rise to a bounded divergence-free potential, it is easier to show that the first term in Eq. (17) is bounded and reproduces correctly the vorticity distribution at the flame front. This is done in Appendix. Thus,

$$K(x, y, \tilde{x}, t) \stackrel{\circ}{=} \frac{i}{2} \int_{\tau_+}^{\tau_-} d\tau M(\tilde{x}, t - \tau). \quad (18)$$

Inserting Eq. (18) into Eq. (11) yields finally

$$v_i^v \stackrel{\circ}{=} \frac{i}{4} \varepsilon_{ik} \partial_k \int_{-A}^{+A} d\tilde{x} \int_{\tau_-}^{\tau_+} d\tau M(\tilde{x}, t - \tau). \quad (19)$$



Since  $\tau_+^* = \tau_-$ , the right hand side of this relation is real.

To make the meaning of the calculation performed more vivid qualitatively, it is useful to mention an interrelation between the roles the conditions a)–c) play in the above derivation. When calculating the near-the-front value of the kernel  $K(x, y, \tilde{x}, t)$ , we retain terms of the first order with respect to  $\tau$ . This is sufficient for the calculation of the vortex component of velocity at the front, taking into account that this quantity is determined by the first spatial derivatives of the kernel, and that  $r = \tau v_+$  at the point defined by Eq. (12). This means, in particular, that  $\tau$  in the function  $M(\tilde{x}, t - \tau)$  cannot be neglected. As was mentioned above, this function cannot be expanded in  $\tau$  either without violating c): the condition  $M(\tilde{x}, t) \rightarrow 0$  for  $t \rightarrow -\infty$  guarantees convergence of the  $\tau$ -integral in the limit  $T \rightarrow \infty$ . At the same time, it is seen from Eq. (19) that as a result of the  $\tau$ -integration, dependence of the function  $M(\tilde{x}, t - \tau)$  on  $\tau$  is transmuted into coordinate dependence. This dependence does not affect the vorticity distribution along the front, because  $\tau = r/v_+$  along the streamlines, so that  $\tau = 0$  when the observation point is taken on the front ( $r = 0$ ). Thus, the seemingly innocent condition c) entails a nontrivial change in the structure of the potential component of the burnt gas velocity in comparison with the steady regime. In the latter case, on the other hand, the condition  $M(\tilde{x}, t) \rightarrow 0$  for  $t \rightarrow -\infty$  does not apply, but since the  $M$ -function is independent of time, it remains independent of the coordinates  $(x, y)$  at all stages of the calculation. Proceeding then as in Ref. [12], one can verify that the divergent contribution to the velocity field, coming from the integration over large  $\tau$ , is also coordinate-independent, so that the condition c) is still met. Finally, it is not difficult to show that the expression (19) cannot be further simplified following the lines of Ref. [12] by omitting the additional potential contribution *after* the spatial differentiation: it turns out that this contribution satisfies the condition a) only in the steady case. We shall return to this point later in Sec. V A.

### A. Definition of the vorticity mode

Having obtained an explicit expression for the vortex component of the burnt gas velocity for a finite  $\Sigma$ , we have to consider the question of the transition to the limits  $A \rightarrow \infty$ ,  $T \rightarrow \infty$ . Generally, the rule these limits are to be taken depends on the problem under consideration. In the case of unsteadily propagating flames, this issue is complicated by the fact that the expression found for the vortex component is essentially nonlocal, both in space and time. The latter nonlocality shows itself explicitly through the  $\tau$ -integration in Eq. (19), and is to be expected from the very outset. In fact, appearance of time non-locality is *inevitable*, because there must exist some mechanism transferring the influence of the flame history onto its current state. Such mechanism is unnecessary only in the stationary case, where all information about the flame past is in a sense left at the infinity downstream. Furthermore, we have seen that the time dependence of the function  $M(\tilde{x}, t)$  is partially transmuted into coordinate dependence, so the time non-locality entails naturally an essential spatial non-locality. This is again in contrast with the steady case, where it turned out to be possible to find a local on-shell expression for the vortex component [Cf. Eq. (37) in 12].

Although the parameter  $T$  does not appear explicitly in Eq. (19), it should be kept in mind that in the course of derivation of this expression, the time dependence of the  $M(\tilde{x}, t)$  function has transmuted into dependence on the spatial coordinates. Hence, in order to preserve property c) of the potential component, the limit  $T \rightarrow \infty$  is generally to be taken under assumption of vanishing of the function  $M(\tilde{x}, t)$  for  $t \rightarrow -\infty$ . As to the limit  $A \rightarrow \infty$ , any such assumption would be irrelevant because of the flame periodicity along the  $x$ -axis. To ensure convergence in this case, we introduce an intermediate regularization of the  $x$ -integral replacing Eq. (19) by

$$v_i^v(\mu) \stackrel{\circ}{=} \frac{i}{4} \varepsilon_{ik} \int_{-A}^{+A} d\tilde{x} e^{-\mu r} \frac{\partial}{\partial x_k} \int_{\tau_-}^{\tau_+} d\tau M(\tilde{x}, t - \tau), \quad (20)$$

where  $\mu > 0$  is a sufficiently large parameter. We then take the limit  $A \rightarrow \infty$  and *define* the vorticity mode as *the analytic* continuation of (20) to the value  $\mu = 0$  along the real axis in the complex  $\mu$ -plane. Replacing also the symbol  $\stackrel{\circ}{=}$  by the equality sign, the definition thus reads

$$v_i^v = \frac{i}{4} \varepsilon_{ik} \left\{ \int_{-\infty}^{+\infty} d\tilde{x} e^{-\mu r} \frac{\partial}{\partial x_k} \int_{\tau_-}^{\tau_+} d\tau M(\tilde{x}, t - \tau) \right\}_{\mu=0^+}. \quad (21)$$

Let us show that the given definition using analytic continuation in  $\mu$  respects the properties a) – c). Rewriting

Eq. (21) as

$$v_i^v = \frac{i}{4}\varepsilon_{ik} \int_{-A_0}^{+A_0} d\tilde{x} \frac{\partial}{\partial x_k} \int_{\tau_-}^{\tau_+} d\tau M(\tilde{x}, t - \tau) \\ + \frac{i}{4}\varepsilon_{ik} \left\{ \left[ \int_{-\infty}^{-A_0} + \int_{+A_0}^{+\infty} \right] d\tilde{x} e^{-\mu r} \frac{\partial}{\partial x_k} \int_{\tau_-}^{\tau_+} d\tau M(\tilde{x}, t - \tau) \right\}_{\mu=0+},$$

where  $A_0 > 0$  is arbitrary, and comparing with Eq. (19) one sees that taking the limit  $A \rightarrow \infty$  followed by the analytic continuation in  $\mu$  does not change the vorticity distribution in the arbitrarily large domain  $|x| < A_0$ . Therefore, the above analytical operations amount to addition of some potential field, so that b) is met. Furthermore, this field is bounded in the sense of c). To see this, consider the quantity

$$\frac{\partial}{\partial x_k} \int_{\tau_-}^{\tau_+} d\tau M(\tilde{x}, t - \tau). \quad (22)$$

Using Eq. (15) and the Newton-Leibnitz formula, this expression is a combination of the functions  $M(\tilde{x}, t - \tau_{\pm})$  times spatial derivatives of  $\tau_{\pm}$ . It follows from Eq. (15) that  $|\tau_{\pm}| = r/\bar{v}_{\pm} \rightarrow \infty$  for  $|\tilde{x}| \rightarrow \infty$ . Hence, if we assume that  $M(\tilde{x}, t)$  is exponentially bounded in a vicinity of the point  $t = \infty$  in the complex  $t$ -plane, *i.e.*,  $|M(\tilde{x}, t)| < e^{c|t|}$  for some  $c > 0$  and  $|t| \rightarrow \infty$ , then there exists a large enough  $\mu$  such that the  $\tilde{x}$ -integral in Eq. (21) converges. On the other hand, the function  $M(\tilde{x}, t)$  is periodic with respect to  $\tilde{x}$  as the result of the flow periodicity. Therefore, all singularities of the expression in the curly brackets in Eq. (21) are off the real axis in the complex  $\mu$ -plane, except possibly for a simple pole at  $\mu = 0$ . The latter corresponds to an additive constant,  $B_k$ , in the quantity (22). The appearance of such a term is not forbidden by the requirement of periodicity in  $\tilde{x}$ .  $B_k$  is independent of  $\mathbf{x}$  by virtue of the same flow periodicity. But after the  $\tilde{x}$ -integration this term gives rise to a contribution of the form  $B_k/\mu + h_k(x)$ , where  $h_k(x)$  vanishes for  $\mu \rightarrow 0$ . Since  $B_k/\mu$  disappears upon spatial differentiation,  $Dv_i^v$  can be continued analytically to  $\mu = 0$ , so the property c) is fulfilled indeed. Finally, the divergence of the vorticity mode

$$\text{div } \mathbf{v}^v = -\frac{i}{4} \left\{ \mu \int_{-\infty}^{+\infty} d\tilde{x} e^{-\mu r} \frac{r_i}{r} \varepsilon_{ik} \frac{\partial}{\partial x_k} \int_{\tau_-}^{\tau_+} d\tau M(\tilde{x}, t - \tau) \right\}_{\mu=0+}$$

is proportional to  $\mu$ . On the other hand, one has  $r_i/r = -\text{sign}(\tilde{x})\delta_{1i} + O(1/|\tilde{x}|)$ . Therefore, the only term contributed by the integral, that survives after continuation to  $\mu = 0$ , is an  $\mathbf{x}$ -independent constant proportional to  $(\delta_{1i}\varepsilon_{ik}B_k/\mu)$ . Thus,  $\text{div } \mathbf{v}^v = \text{const}$ , and the condition a) is satisfied.

#### IV. CLOSED DESCRIPTION OF NON-STATIONARY FLAMES

We are now in position to write down an integro-differential equation relating the on-shell values of the fuel velocity and the front position function. Let us introduce the complex variable  $z = x + iy$ , and the complex velocity  $\omega = u + iw$ . By virtue of the properties a), b) the complex quantity  $d\omega^p/dz$ , where  $\omega^p = u^p + iw^p$ , is an analytical function of the complex variable  $z$  in the downstream region. In conjunction with the property c), analyticity of  $d\omega^p/dz$  can be expressed in the form of the following dispersion relation [11, 12]

$$(1 + i\hat{\mathcal{H}})(\omega_+^p)' = 0, \quad (23)$$

where the prime denotes  $x$ -differentiation, and the action of the operator  $\hat{\mathcal{H}}$  on an arbitrary function  $a(x)$  is defined by

$$(\hat{\mathcal{H}}a)(x) = \frac{1 + if'(x, t)}{\pi} \oint_{-\infty}^{+\infty} d\tilde{x} \frac{a(\tilde{x})}{\tilde{x} - x + i[f(\tilde{x}, t) - f(x, t)]}, \quad (24)$$



slash denoting the principle value of the integral.  $\hat{\mathcal{H}}$  has properties similar to the Hilbert operator  $\hat{H}$  (and is effectively the Hilbert transform along the front). In particular, it was proved by [12] that

$$\hat{\mathcal{H}}^2 = -1. \quad (25)$$

The identity (23) relates in a complicated way the on-shell values of the burnt gas velocity and the flame front position. The fuel velocity also satisfies the conditions a) – c), this time in the upstream region. Indeed, a) is just the differentiated Eq. (2), b) follows from the Thomson theorem and the boundary conditions upstream, and c) is true because the fuel velocity is bounded. The consequence of these properties is the following dispersion relation for  $\omega_- = u_- + iw_-$

$$(1 - i\hat{\mathcal{H}})(\omega_-)' = 0. \quad (26)$$

Denote  $[\mathbf{v}]$  the jump of the gas velocity across the flame front,  $[\mathbf{v}] = \mathbf{v}(x, f(x, t) + 0) - \mathbf{v}(x, f(x, t) - 0)$ . Then the sought equation for  $\omega_-, f$  is obtained by substituting

$$\omega_+^p = -\omega_+^v + \omega_- + [\omega]$$

in Eq. (23), and using Eqs. (21), (26)

$$2(\omega_-)' + (1 + i\hat{\mathcal{H}}) \left\{ [\omega] - \frac{i}{4} \int_{-\infty}^{+\infty} d\tilde{x} e_k \frac{\partial}{\partial x_k} \int_{\tau_-}^{\tau_+} d\tau M(\tilde{x}, t - \tau) \right\}' = 0, \quad (27)$$

where  $e_k = \varepsilon_{2k} + i\varepsilon_{1k}$ , and we omit for brevity the regularizing factor  $e^{-\mu r}$  in the integrand as well as the accompanying symbol of analytic continuation. In the last term on the left, the argument  $y$  is understood to be set equal to  $f(x, t)$  after the spatial partial differentiation is performed, but before the  $x$ -differentiation denoted by the prime. The value of vorticity at the front and the normal velocity of the burnt gas, entering the function  $M(\tilde{x}, t - \tau)$ , as well as the velocity jumps at the front are known functionals of the on-shell fuel velocity [18, 19]. For instance, for zero-thickness flame fronts one has

$$\bar{v}_+^n = \theta, \quad [u] = \frac{\theta - 1}{N}, \quad [w] = -f' \frac{\theta - 1}{N}, \quad (28)$$

$$\sigma_+ = -\frac{\theta - 1}{\theta N} \left\{ \frac{Dw_-}{Dt} + f' \frac{Du_-}{Dt} + \frac{1}{N} \frac{Df'}{Dt} \right\}, \quad (29)$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \left( w_- + \frac{f'}{N} \right) \frac{\partial}{\partial x}.$$

Thus, the complex Eq. (27) gives two equations for three functions  $w_-(x, t)$ ,  $u_-(x, t)$  and  $f(x, t)$ . Together with the evolution equation

$$(\bar{\mathbf{v}}_- \cdot \mathbf{n}) = 1 + S(u_-, w_-, f'), \quad (30)$$

where  $S$  is a known functional of its arguments, proportional to the flame front thickness (or, rather, the cut-off wavelength  $\lambda_c$ ), Eq. (27) provides one with a closed description of unsteady flames in the most general form. Its application to various particular problems is given in the next section.

Before proceeding, the following point is worth to comment on. In the analysis that led us to (27), the front was represented by a graph  $y = f(x, t)$ , which excludes the overhangs or the fronts that double back on themselves. This over-restrictive assumption, adopted so far for simplicity, can be relaxed as follows. Let us parameterize the flame front by a real parameter  $\xi$  so that  $(x(\xi), y(\xi))$  be a diffeomorphic mapping of the interval  $-\infty < \xi < +\infty$  onto the front at time instant  $t$ . Then, setting  $x(\xi) + iy(\xi) = Z(\xi, t)$ , we define the new metric coefficient  $N(\xi, t)$  in terms of the infinitesimal arclength along the front,  $dl$ , by  $dl = N(\xi, t) d\xi$ ; in other words,  $N(\xi, t) = |\partial_\xi Z(\xi, t)|$ , which is nowhere singular for smooth fronts. As long as the fuel and the burnt-gas regions remain connected domains, the Cauchy theorem guarantees the existence of a generalized operator  $\hat{\mathcal{H}}$  such that  $(1 \pm i\hat{\mathcal{H}})d\omega_\pm/d\xi = 0$ . Specifically, when acting on a smooth  $a(\xi, t)$ , the new  $\hat{\mathcal{H}}$  produces

$$(\hat{\mathcal{H}}a)(\xi, t) = \frac{\partial_\xi Z}{\pi} \oint \frac{a(\tilde{\xi}, t) d\tilde{\xi}}{Z(\tilde{\xi}, t) - Z(\xi, t)}, \quad (31)$$

instead of (24). Accordingly, if  $M(\xi, t)$  is still defined as  $M = N(\xi, t) \sigma_+(\xi, t) \bar{u}_+^n(\xi, t)$ , Eq. (27) is formally unchanged, except that the prime now denotes  $d/d\xi$ . Of course, the time derivatives, now at fixed  $\xi$ , must be handled in a way consistent with the new representation of the flame front. Yet such a re-parameterization does not capture situations when isolated pockets of unburnt fuel form, because the fresh domain then ceases to be path-connected. Unfortunately, as long as a *local* propagation law [Cf. (30)] is employed such a phenomenon cannot be excluded *a priori*: in no way can a flame element “know” that another one is to produce a “head-on” collision.

The above re-parameterization is unnecessary for the wrinkled fronts considered below.

## V. EQUATIONS (27), (30) IN LIMITING CASES

To give a consistency check for Eq. (27) and also to gain a deeper insight into the structure of this equation, we use it below to derive anew classical results on flame front dynamics.

### A. Darrieus-Landau instability of zero-thickness flames

Let us consider first the classical linear stability problem of zero-thickness planar flame propagation [5, 6]. In this case,  $\bar{v}_+^n = \theta$ ,  $N = 1$ , while the linearized on-shell vorticity (29) reads

$$\sigma_+ = -\frac{\theta - 1}{\theta} \left( \frac{\partial w_-}{\partial t} + \frac{\partial^2 f}{\partial t \partial x} \right).$$

Accordingly, expression (21) simplifies to

$$v_i^v = -\frac{i(\theta - 1)}{4} \varepsilon_{ik} \left\{ \int_{-\infty}^{+\infty} d\tilde{x} e^{-\mu|x-\tilde{x}|} \frac{\partial}{\partial x_k} \int_{\tau_-}^{\tau_+} d\tau \left( \frac{\partial w_-}{\partial t} + \frac{\partial^2 f}{\partial t \partial x} \right) \right\}_{\mu=0^+}. \quad (32)$$

Since the integrand here is a first order quantity, it is sufficient to calculate  $\tau_{\pm}$  for a plane front which is assumed to be at  $y = 0$

$$\tau_{\pm} = \frac{y}{\theta} \pm i \frac{|x - \tilde{x}|}{\theta}. \quad (33)$$

Let the disturbance be periodic in  $x$ , growing exponentially with time. Spatial periodicity of the linear problem is most conveniently represented in the complex form, in which case

$$f(x, t), u_-(x, t), w_-(x, t) \sim e^{ikx + \nu t}. \quad (34)$$

However, it should be kept in mind that the coefficients in Eq. (27) are also complex. In order to preserve the right complex structure of this equation, all the functions involved are to be written in the form containing no imaginary coefficients. To find  $v_i^v$ , we have to evaluate the following integrals

$$I_k(x, y, t, \mu) = \int_{-\infty}^{+\infty} d\tilde{x} e^{-\mu|x-\tilde{x}|} \frac{\partial}{\partial x_k} \int_{y/\theta - i|x-\tilde{x}|/\theta}^{y/\theta + i|x-\tilde{x}|/\theta} d\tau e^{ik\tilde{x} + \nu(t-\tau)}$$

for  $\mu > 0$  and  $k = 1, 2$ . A straightforward calculation gives

$$\begin{aligned} I_1(x, y, t, \mu) &= \frac{i}{\theta} e^{ikx + \nu t - \nu y/\theta} \left\{ \left( \frac{1}{\mu + ik + i\nu/\theta} + \frac{1}{-\mu + ik + i\nu/\theta} \right) + (\nu \rightarrow -\nu) \right\}, \\ I_2(x, y, t, \mu) &= \frac{1}{\theta} e^{ikx + \nu t - \nu y/\theta} \left\{ \left( \frac{1}{\mu + ik + i\nu/\theta} + \frac{1}{-\mu + ik + i\nu/\theta} \right) - (\nu \rightarrow -\nu) \right\}. \end{aligned}$$

where  $(\nu \rightarrow -\nu)$  is shorthand for the preceding parenthesis with  $\nu$  changed into its opposite. The on-shell values of these functions analytically continued to  $\mu = 0$  are

$$I_1(x, 0, t, 0) = \frac{4}{\theta} e^{ikx + \nu t} \frac{k}{k^2 - \nu^2/\theta^2}, \quad I_2(x, 0, t, 0) = \frac{4i}{\theta} e^{ikx + \nu t} \frac{\nu/\theta}{k^2 - \nu^2/\theta^2}. \quad (35)$$

Using this in Eq. (32) yields

$$w_+^v = \frac{\nu/\theta}{k^2 - \nu^2/\theta^2} \frac{(\theta - 1)}{\theta} (\nu w_- + ik\nu f), \quad (36)$$

$$u_+^v = \frac{ik}{k^2 - \nu^2/\theta^2} \frac{(\theta - 1)}{\theta} (\nu w_- + ik\nu f). \quad (37)$$

To put these expressions in an explicitly real form, it is sufficient to write

$$w_+^v = -\frac{\sigma_+ \nu/\theta}{k^2 - \nu^2/\theta^2}, \quad u_+^v = -\frac{\sigma'_+}{k^2 - \nu^2/\theta^2}.$$

Next, we note that in the linear approximation, the operator  $\hat{\mathcal{H}}$  becomes just the Hilbert operator  $\hat{H}$ , whose action on the harmonic functions is defined by

$$\hat{H} \exp(ikx) = i\chi(k) \exp(ikx), \quad k \neq 0, \quad (38)$$

where

$$\chi(k) = \begin{cases} +1, & k > 0, \\ 0, & k = 0, \\ -1, & k < 0. \end{cases}$$

Equation (27) thus becomes

$$2(\omega_-)' + (1 + i\hat{H}) \left\{ [\omega] + \frac{\sigma'_+ + i\sigma_+ \nu/\theta}{k^2 - \nu^2/\theta^2} \right\}' = 0. \quad (39)$$

Extracting the real and imaginary parts of this equation, we find

$$2(u_-)' + \left\{ [u] - \hat{H}[w] + \frac{\sigma'_+ - \hat{H}\sigma_+ \nu/\theta}{k^2 - \nu^2/\theta^2} \right\}' = 0, \quad (40)$$

$$(w_-)' = \hat{H}(u_-)'. \quad (41)$$

Finally, upon linearization the jump conditions (28) simplify to

$$[u] = \theta - 1, \quad [w] = -f'(\theta - 1), \quad (42)$$

while the linearized evolution equation reads

$$u_- - \frac{\partial f}{\partial t} = 1. \quad (43)$$

Inserting these into Eq. (40) and then substituting  $\sigma_+ = -(\theta - 1)/\theta (i\chi(k)\nu^2 f + ik\nu f)$  leads after some simple algebra to the equation

$$\frac{\theta + 1}{\theta} \nu^2 + 2\nu|k| - (\theta - 1)k^2 = 0, \quad (44)$$

which is nothing but the famous Darrieus-Landau dispersion relation determining the perturbation growth rate as a function of the wave number and the gas expansion coefficient [5, 6]

$$\nu = \frac{\theta}{\theta + 1} \left( \sqrt{1 + \theta - \frac{1}{\theta}} - 1 \right) |k|. \quad (45)$$

The effects related to the finite front thickness can be explicitly accounted for in the above linear analysis by including terms linear in  $f''$  in the right hand side of Eqs. (42), (43). This would change the last term in Eq. (44) by the extra factor  $(1 - |k| \lambda_c/2\pi)$ , and make  $\nu$  roughly parabolic for all  $|k| \leq 2\pi/\lambda_c$ ;  $\max(\nu)$  enables one to estimate the typical growth time as  $t_{DL} \simeq 2\lambda_c/\pi(\sqrt{\theta} - 1)$ , to be used later (Sec. 6).

Next, let us return to the remark made after Eq. (14). It was mentioned there that even if one is interested only in the on-shell values of the vorticity component, the  $\tau$ -dependence of the function  $M(\tilde{x}, t - \tau)$  cannot be neglected. Evidently, doing so amounts simply to rewriting the equation obtained by Kazakov [11, 12] for steady flames in terms of the local current flow velocity relative to the front. It is not difficult to verify that in the case under consideration, this would change the coefficient of  $\nu^2$  in Eq. (44) to the wrong value  $(\theta - 1)/\theta$ . This change is thus a reflection of the memory effects encoded in the function  $M(\tilde{x}, t - \tau)$ . Equation (27) properly takes into account these effects, correctly reproducing the Darrieus-Landau relation, and automatically captures all the aspects of flame dynamics related to inertia.

At this stage of the analysis it is appropriate to pause and discuss the meaning of the analytic continuation appearing in the definition (21) in somewhat more detail. This continuation is used to make the  $\tilde{x}$ -integral meaningful in the limit  $A \rightarrow \infty$ . One can avoid using this analytic means if the formal result of improper integration along the infinite flame front is treated in the sense of distributions. Indeed, in this case expressions (35) for the vortex component would contain additional terms proportional to  $\delta(k + \nu/\theta)$  or  $\delta(k - \nu/\theta)$  coming from the integration over large  $\tilde{x}s$ . To see that these terms are inconsequential we recall that the above consideration of the single  $k$ -mode evolution is not completely adequate from the physical point of view, because in practice one always deals with wave packets consisting of a continuum of wave numbers  $k$ , rather than with a single mode. This means that the physical expression for the flame front position with the given  $k_0$  is obtained by integrating the found solution  $f(x, t)$  with some weight over a small but finite range  $\Delta k$  around  $k_0$ . Upon this integration all terms involving  $\delta(k \pm \nu/\theta)$  disappear, because  $\nu = \pm k\theta$  are not roots of the Darrieus-Landau relation. The two procedures are thus equivalent.

### B. The small $(\theta - 1)$ expansion

Let us next consider the case of small gas expansion. We will verify that within the framework of the asymptotic expansion with respect to  $\theta - 1 \equiv \alpha$ , Eq. (27) reduces at the first post-Sivashinsky order to the well-known Sivashinsky-Clavin equation [14]<sup>3</sup>. To perform the asymptotic expansion we recall that the cutoff wavelength for the short wavelength perturbations  $\lambda_c$  is of the order  $1/\alpha$ . This means that the wave numbers involved are  $O(\alpha)$ . In other words, spatial differentiation of a flow variable raises its order by one; in particular,  $f' = O(\alpha)$ . Also, Eq. (45) tells us that for small  $\alpha$ , the perturbation growth rate  $\nu = k\alpha/2 = O(\alpha^2)$ , so the order of a flow variable is raised by two upon time differentiation. It follows then from Eq. (30) [with  $S \equiv 0$ ] that  $u_- = 1 + O(\alpha^2)$ , while potentiality of the upstream flow implies that  $w_- = O(\alpha^2)$  [this is clearly seen from the dispersion relation (26)]. The first post-Sivashinsky approximation corresponds to retaining terms of the fourth order in Eq. (27), or equivalently,  $O(\alpha^3)$ -terms before the spatial differentiation. It was shown by [12] that to this order,  $\hat{\mathcal{H}}$  becomes just the Hilbert operator. Therefore, like in the linear case considered above, the real and imaginary parts of Eq. (27) are readily separated to give

$$2(u_-)' + \left\{ [u] - \hat{H}[w] + (\varepsilon_{1k}\hat{H} - \varepsilon_{2k})\frac{i}{4} \int_{-\infty}^{+\infty} d\tilde{x} \frac{\partial}{\partial x_k} \int_{\tau_-}^{\tau_+} d\tau M(\tilde{x}, t - \tau) \right\}' = 0, \quad (46)$$

$$(w_-)' = \hat{H}(u_-)'. \quad (47)$$

Since the quantity  $J_k \equiv \partial_k \int_{\tau_-}^{\tau_+} d\tau M$  is under the sign of spatial integral, we need the function  $M(\tilde{x}, t)$  to the fourth order in  $\alpha$ . To this accuracy, it is equal to the on-shell vorticity

$$M = \sigma_+ = -(\theta - 1) \left( \dot{f}' + f' f'' \right),$$

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<sup>3</sup> It is worth noting that the small expansion parameter used by Sivashinsky and Clavin is  $\gamma \equiv (\theta - 1)/\theta = \alpha/(1 + \alpha)$ , rather than  $\alpha$ . The reason for switching from  $\alpha$  to  $\gamma$ , found sometimes in the literature, is that the latter would improve the expansion accuracy/convergence. The argument given in this connection, namely, that  $\gamma < 1$  for all  $\theta$ , while  $\alpha$  is small only for  $\theta$  close to 1, is not quite correct. Validity of an expansion is determined not by the value of the expansion parameter, but by the relative value of the terms neglected in the course of the calculation. It is true that expanding the decreasing function  $f(\theta) = 1/\theta = 1/(1 + \alpha)$  in terms of  $\alpha/(1 + \alpha)$  instead of  $\alpha$  is an improvement. However, it is not in the case of the increasing function  $f(\theta) = \theta - 1 = \alpha$ , which certainly appears in the governing equations [see Eqs. (27), (28)]. Thus, whether a change of the expansion parameter improves the expansion accuracy/convergence is a question of the structure of the whole perturbation series, which cannot be resolved from the knowledge of its first few terms. It is one of the goals of our approach to make questions of this kind accessible for theoretical analysis.

the dot denoting differentiation with respect to the time  $t$ . As this quantity is already  $O(\alpha^4)$ ,  $\tau_{\pm}$  can be taken in the form (33) with  $\theta = 1$ . Thus, performing the spatial differentiation and setting  $y = 0$  gives

$$\begin{aligned} J_1 &= i\chi(x - \tilde{x}) \{ \sigma_+(\tilde{x}, t - i|x - \tilde{x}|) + \sigma_+(\tilde{x}, t + i|x - \tilde{x}|) \} , \\ J_2 &= \{ \sigma_+(\tilde{x}, t - i|x - \tilde{x}|) - \sigma_+(\tilde{x}, t + i|x - \tilde{x}|) \} . \end{aligned}$$

Let us show that imaginary parts in the argument of  $\sigma_+$  can be omitted. Note that within the asymptotic expansion in  $\alpha$ , dependence of  $\sigma_+$  on  $i|x - \tilde{x}|$  can be treated perturbatively. Indeed, let us write formally

$$\sigma_+(\tilde{x}, t - i|x - \tilde{x}|) = \sigma_+(\tilde{x}, t) - i|x - \tilde{x}| \dot{\sigma}_+(\tilde{x}, t) - \frac{1}{2}(x - \tilde{x})^2 \ddot{\sigma}_+(\tilde{x}, t) + \dots$$

To assess the relative order of consecutive terms in this series, we have first to get rid of the factors containing explicit coordinate dependence, which can be done by successive integration by parts with respect to  $\tilde{x}$ . One has, for example,

$$\begin{aligned} \int_{-\infty}^{+\infty} d\tilde{x} |x - \tilde{x}| \dot{\sigma}_+(\tilde{x}, t) &= \int_{-\infty}^{+\infty} d \left( \int^{\tilde{x}} dx_1 \dot{\sigma}_+(x_1, t) \right) |x - \tilde{x}| , \\ &= \int_{-\infty}^{+\infty} d\tilde{x} \chi(x - \tilde{x}) \left( \int^{\tilde{x}} dx_1 \dot{\sigma}_+(x_1, t) \right) , \end{aligned}$$

where according to the discussion in the preceding section, the contributions of infinitely remote parts of the front have been omitted. Since each time differentiation adds two powers of  $\alpha$ , while spatial integration subtracts only one, we conclude that the above expansion for  $\sigma_+$  is effectively an asymptotic series in powers of  $\alpha$ . Hence, to the fourth order in  $\alpha$

$$J_1 = 2i\chi(x - \tilde{x})\sigma_+(\tilde{x}, t), \quad J_2 = 0,$$

and so

$$\begin{aligned} \int_{-\infty}^{+\infty} d\tilde{x} J_1 &= -2i(\theta - 1) \int_{-\infty}^{+\infty} d\tilde{x} \chi(x - \tilde{x}) \left( \dot{f}' + f' f'' \right) (\tilde{x}, t) \\ &= -4i(\theta - 1) \int_{-\infty}^{+\infty} d\tilde{x} \delta(x - \tilde{x}) \left( \dot{f} + \frac{f'^2}{2} \right) (\tilde{x}, t) \\ &= -4i(\theta - 1) \left( \dot{f} + \frac{f'^2}{2} \right) (x, t), \end{aligned}$$

where the boundary terms at  $\tilde{x} = \pm\infty$  were omitted again in the course of the integration by parts, and the relation  $\partial\chi(x - \tilde{x})/\partial\tilde{x} = -2\delta(x - \tilde{x})$  was used. Inserting this together with the velocity jumps into Eqs. (46), (47) yields within the given accuracy

$$2u_- + (\theta - 1) \left( \hat{H} f' + \dot{f} \right) = C_1(t), \quad (48)$$

$$2w_- + (\theta - 1) \left( -f' + \hat{H} \dot{f} \right) = C_2(t), \quad (49)$$

where  $C_{1,2}(t)$  are two “integration constants” (some  $x$ -independent functions of time). Since the left hand side of (49) is odd in  $x$ , one has  $C_2 \equiv 0$ . Using these formulas in the evolution equation (30) written in the form

$$u_- - \dot{f} - f' w_- = 1 + \frac{f'^2}{2}$$

leads to the equation for the function  $f(x, t)$

$$\frac{\theta + 1}{2} \dot{f} + \frac{\theta}{2} f'^2 + C(t) = -\frac{\theta - 1}{2} \hat{H} f', \quad (50)$$

where  $C(t) = 1 - C_1(t)/2$ . The function  $C(t)$  can be found by averaging the obtained equation along the flame front:

$$C(t) = -\frac{\theta + 1}{2} \langle \dot{f} \rangle - \frac{\theta}{2} \langle f'^2 \rangle.$$

Equation (50) is nothing but the Sivashinsky-Clavin equation [14], with the term  $C(t)$  added according to [20]. Account of the transport processes inside the front would have added a term proportional to  $\lambda_c f''$  to the right hand side of Eq. (50).

Finally, it is interesting to note that the memory effects are insignificant not only at the lowest order of the small  $(\theta - 1)$ -expansion, but also at the first post-Sivashinsky approximation considered here. Namely, a direct calculation shows that replacing  $M(\tilde{x}, t - \tau)$  by  $M(\tilde{x}, t)$  in Eq. (27) does not change Eq. (50). This is natural, because the condition  $\alpha \ll 1$  implies slow dynamics. However, this is not the case already in the next order, which is possibly one reason why the memory effects are often overlooked.

### C. Flame propagation in time-dependent gravitational field

Depending on the direction of flame propagation, gravitational field leads either to strengthening or damping of the Darrieus-Landau instability. Also, the influence of sound waves on the front dynamics can be described effectively as the flame propagation in a time-dependent gravitational field [13]. Let  $g(t)$  denote its strength, with the convention that  $g > 0$  corresponds to a stabilizing effect [see Eq. (54)]. Inclusion of the gravitational field does not change the flow equations (2), (3), so that their consequence, Eq. (27), has the same structure. The jump conditions (28) for the gas velocity are also left intact by gravity. The only place where  $g(t)$  appears in our approach is the expression of the on-shell vorticity (as a result of baroclinic effects inside the front). Namely, the gravity-induced jump in this quantity to be added to the right hand side of Eq. (29) is [21]<sup>4</sup>

$$\Delta\sigma_+ = -\frac{(\theta - 1)}{\theta N} g(t) f'(x, t). \quad (51)$$

If development of the Darrieus-Landau instability is suppressed by the gravitational field, then it is natural to consider harmonic front perturbations instead of the exponentially growing ones used in Sec. V A. The choice between the two representations is just a matter of convenience. Namely, it will be seen below that although the intermediate procedure of analytic continuation in  $\mu$  is somewhat different for  $\nu$  imaginary, the final equations for the front position are simply analytic continuations of one another with respect to frequency.

Because of the time dependence of  $g(t)$ , the function  $f(x, t)$  is now to be taken as a superposition of an arbitrary number of harmonics

$$f(x, t) = \int_{-\infty}^{+\infty} d\omega dk f(k, \omega) e^{ikx - i\omega t}. \quad (52)$$

It is also convenient to introduce the Fourier decomposition of the function  $G(x, t) = g(t) f'(x, t)$

$$G(x, t) = \int_{-\infty}^{+\infty} d\omega dk G(k, \omega) e^{ikx - i\omega t}. \quad (53)$$

To perform the analytic continuation with respect to  $\mu$  in Eq. (21), we will assume that  $f(k, \omega)$ ,  $G(k, \omega)$ , considered as functions of  $\omega$ , vanish outside a large but finite frequency band  $|\omega| \leq \omega_0$ . Then, choosing  $\mu > \omega_0/\theta$ , inserting the above Fourier decompositions in the linearized expression for the on-shell vorticity, performing  $\tilde{x}$ -integration in Eq. (21) as before, and continuing the result analytically to  $\mu = 0$ , we find

$$\begin{aligned} w_+^v &= -i \frac{\theta - 1}{\theta} \int_{-\infty}^{+\infty} d\omega dk e^{ikx - i\omega t} \frac{\omega/\theta}{k^2 + \omega^2/\theta^2} \{(-i\chi(k)\omega^2 + k\omega)f(k, \omega) + G(k, \omega)\}, \\ u_+^v &= i \frac{\theta - 1}{\theta} \int_{-\infty}^{+\infty} d\omega dk e^{ikx - i\omega t} \frac{k}{k^2 + \omega^2/\theta^2} \{(-i\chi(k)\omega^2 + k\omega)f(k, \omega) + G(k, \omega)\}. \end{aligned}$$

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<sup>4</sup> More generally,  $\Delta\sigma_+ = \frac{(\theta-1)}{\theta N} \{\Gamma(x, f(x, t), t)\}'$ , where  $\Gamma(x, y, t)$  is the gravitational potential,  $\mathbf{g} = -\nabla\Gamma$ .



The first terms on the right of these equations are just analytic continuations of the expressions (36), (37) to the imaginary value of the growth rate:  $\nu \rightarrow -i\omega$ . As in the Darrieus-Landau problem, Eq. (27) splits into two real equations

$$2(u_-)' + \left\{ [u] - \hat{H}[w] - u_+^v + \hat{H}w_+^v \right\}' = 0, \quad (w_-)' = \hat{H}(u_-)',$$

which after substitution of the found expressions for  $v_i^k$  together with Eqs. (42), (43) yield

$$\int_{-\infty}^{+\infty} d\omega dk e^{ikx - i\omega t} \left\{ \left[ \frac{\theta+1}{\theta} \omega^2 + 2i\omega|k| + (\theta-1)k^2 \right] kf(k, \omega) + \frac{\theta-1}{\theta} ik\chi(k)G(k, \omega) \right\} \left( \frac{\omega}{\theta} + ik \right)^{-1} = 0.$$

Acting on this equation by the operator

$$\left( \frac{i}{\theta} \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) \int dx,$$

and taking into account definitions (52), (53), we arrive at the well-known equation for the front position [13]

$$\frac{\theta+1}{\theta} \ddot{f} - 2\hat{H}\dot{f}' + (\theta-1)f'' - \frac{\theta-1}{\theta} g(t)\hat{H}f' = C(t), \quad (54)$$

where  $C(t)$  is a function of time appearing as the result of the spatial integration symbolized by  $\int dx$ . As before,  $C(t)$  can be found by averaging Eq. (54) along the front. Equation (54) is well-known to be the key ingredient in understanding the parametric flame response to oscillating  $g(t)$ s [10, 16], in which case the inertia (hence, memory) effects play the central role.

#### D. Steady flame propagation

In the case of stationary flames, the  $M$ -function is time-independent, whence the  $\tau$ -integration in Eq. (27) is trivial. One has

$$\begin{aligned} \frac{i}{4} \int_{-\infty}^{+\infty} d\tilde{x} e_k \frac{\partial}{\partial x_k} \int_{\tau_-}^{\tau_+} d\tau M(\tilde{x}, t - \tau) &= -\frac{1}{2} \int_{-\infty}^{+\infty} d\tilde{x} \frac{M(\tilde{x})}{v_+} e_k \frac{\partial}{\partial x_k} \sqrt{r^2 - \frac{(\mathbf{r} \cdot \mathbf{v}_+)^2}{v_+^2}}, \\ &= -\frac{1}{2} \int_{-\infty}^{+\infty} d\tilde{x} M(\tilde{x}) \frac{e_k \beta_k}{v_+}, \end{aligned}$$

where

$$\beta_k = \left( \frac{r_k}{r} - \Omega \frac{v_{k+}}{v_+} \right) \frac{1}{\sqrt{1 - \Omega^2}}. \quad (55)$$

This vector satisfies

$$\beta_k \beta_k = 1, \quad \beta_k v_{k+} = 0,$$

i.e.,  $\beta$  is the unit vector orthogonal to  $\mathbf{v}_+$ . In addition to that,  $\beta$  changes its sign at the point  $\tilde{x}$  satisfying Eq. (12). Hence, its scalar product with the complex vector  $e_i$  can be written on-shell as

$$e_k \beta_k = -\frac{\omega_+}{v_+} \chi(x - \tilde{x}).$$

Differentiation with respect to  $x$  using the formula  $\chi'(x) = 2\delta(x)$  makes the  $\tilde{x}$ -integration trivial, so that Eq. (27) takes the form

$$2(\omega_-)' + \left( 1 + i\hat{\mathcal{H}} \right) \left\{ [\omega]' - \frac{Nv_+^n \sigma_+ \omega_+}{v_+^2} \right\} = 0 \quad (56)$$

which is exactly Eq. (42) of Ref. [12].

In connection with Eq. (56) one can indicate why viscous effects have virtually no influence on the shape and the speed of steady flames, except possibly a small one on the cut-off wavelength ( $\lambda_c$ ) itself, the only internal reference length of our problem. The Reynolds number based on  $\lambda_c$  and the *burnt* gas properties and speeds is normally about  $10^2$ . The viscous length  $\lambda_{\text{vis}}$  – the distance at which the vorticity present behind the front-crests of transverse size  $\sim \lambda_c$  dissipates noticeably – thus exceeds  $\lambda_c$  by two orders. This is definitely too late (in terms of the Lagrangian time  $\tau$ ) to modify Eq. (56), whose validity only requires that  $\sigma \rightarrow \sigma_+$  in an infinitesimal layer ( $\tau \rightarrow 0^+$ ) downstream of the front ( $\tau = 0$ ). As to unsteady flames the picture is more involved, even though the potential fuel flow is still unaffected directly by viscous effects. The problem would certainly deserve further studies, for one cannot *a priori* exclude viscosity-related large-scale phenomena occurring far from the front, that could nevertheless influence its dynamics indirectly. We merely mention here that the spontaneous flame dynamics is initially little affected, because the Darrieus-Landau time  $t_{DL}$  (see Sec. 5.1) is well shorter than  $t_{\text{vis}} = \lambda_{\text{vis}}/\theta$ : the decay of  $M(\tilde{x}, t - \tau)$  for  $\tau \gg 1$  is fully controlled by the wrinkle growth itself.

## VI. STABILITY ANALYSIS OF GENERAL STEADY FLAME PATTERNS

Let us use Eq. (27) to derive an equation governing propagation of small disturbances along a given steady flame pattern. As in the Darrieus-Landau or stationary problems considered above, Eq. (27) essentially simplifies in this case, because the time non-locality is no longer a complication. Indeed, since there is no external time-varying field (as the stationary regime is assumed to exist) and the flame disturbance is small, it is sufficient to consider perturbations having the form

$$\delta f(x, t) = \tilde{f}(x)e^{\nu t}, \quad \delta w_-(x, t) = \tilde{w}(x)e^{\nu t}, \quad \delta u_-(x, t) = \tilde{u}(x)e^{\nu t},$$

in which case the time dependence is prescribed, and the  $\tau$ -integration in Eq. (27) is readily done. Let us gather the functions  $f(x, t)$ ,  $w_-(x, t)$ ,  $u_-(x, t)$  into an array  $\{\xi_\alpha(x, t)\}$ ,  $\alpha = 1, 2, 3$ :

$$(\xi_1, \xi_2, \xi_3) = (f, w_-, u_-),$$

and denote  $\xi_\alpha^{(0)}(x)$  the given solution of Eq. (56). We have to linearize the left hand side of Eq. (27) with respect to  $\delta \xi(x, t) \equiv \tilde{\xi}(x)e^{\nu t} = (\tilde{f}, \tilde{w}, \tilde{u})e^{\nu t}$ . Depending on whether the function  $M(\tilde{x}, t)$  is varied, one finds contributions of two types. If  $\tilde{\xi}_\alpha$  comes from terms other than  $M$ , then  $M(\tilde{x}, t)$  is evaluated for  $\xi = \xi^{(0)}$  and hence is time-independent. As we saw in the preceding section, Eq. (27) simplifies in this case to Eq. (56), so the corresponding contribution to the variation is given by the left hand side of Eq. (56) linearized around  $\xi^{(0)}$ , with  $M$  kept fixed. The other contribution comes from variation of  $M(\tilde{x}, t)$  and has the form

$$\Delta_M = \left(1 + i\hat{\mathcal{H}}\right) \left\{ -\frac{i}{4} \int_{-\infty}^{+\infty} d\tilde{x} e_k \frac{\partial}{\partial x_k} \int_{\tau_-}^{\tau_+} d\tau \hat{M}_\alpha(\tilde{x}) \tilde{\xi}_\alpha(\tilde{x}) e^{\nu(t-\tau)} \right\}',$$

where  $\hat{M}_\alpha(\tilde{x})$  is the differential operator obtained by linearizing the function  $M(\tilde{x}, t)$  around the stationary solution, and changing  $\partial/\partial t \rightarrow \nu$  afterwards. A straightforward calculation gives

$$\Delta_M = -\frac{e^{\nu t}}{2} \left(1 + i\hat{\mathcal{H}}\right) \int_{-\infty}^{+\infty} d\tilde{x} \hat{M}_\alpha(\tilde{x}) \tilde{\xi}_\alpha(\tilde{x}) \frac{\omega_+}{v_+^2} \left\{ \exp\left(-\frac{\nu r}{v_+} e^{-i\phi}\right) \chi(x - \tilde{x}) \right\}',$$

where  $\phi \in [-\pi, +\pi]$  is the angle between the vectors  $\mathbf{r}$ ,  $\mathbf{v}_+$ , defined positive if the rotation from  $\mathbf{v}_+$  to  $\mathbf{r}$  is clockwise. It is not difficult to verify that the  $x$ -differentiation of the step function in the latter expression gives rise to the term which is just the variation of the left hand side of Eq. (56) under variation of the function  $M(\tilde{x}, t)$ . Furthermore, one has

$$(r \sin \phi)' = \tau_i \frac{\partial(r \sin \phi)}{\partial x_i} = \tau_i \beta_i \chi(x - \tilde{x}),$$

where  $\tau_i = \varepsilon_{ik} n_k$  is the unit vector tangential to the flame front. Taking into account also that

$$\varepsilon_{ik} \beta_i = \frac{v_{k+}}{v_+} \chi(x - \tilde{x}),$$

we find

$$(r \sin \phi)' = \varepsilon_{ik} n_k \beta_i \chi(x - \tilde{x}) = \frac{v_{k+} n_k}{v_+} \equiv \frac{v_+^n}{v_+}.$$

Similarly,

$$(r \cos \phi)' = \frac{v_{k+} \tau_k}{v_+} \equiv \frac{v_+^\tau}{v_+}.$$

Putting all these results together and omitting the factor  $e^{\nu t}$ , we obtain the following equation for the  $x$ -dependent parts of the perturbations

$$\begin{aligned} 2\tilde{\omega}' + \int_{-\infty}^{+\infty} d\tilde{x} \tilde{\xi}_\alpha(\tilde{x}) \frac{\delta}{\delta \xi_\alpha(\tilde{x})} \left(1 + i\hat{\mathcal{H}}\right) \left\{ [\omega]' - \frac{N v_+^n \sigma_+ \omega_+}{v_+^2} \right\} \\ = \frac{i\nu}{2} \left(1 + i\hat{\mathcal{H}}\right) \int_{-\infty}^{+\infty} d\tilde{x} \hat{M}_\alpha(\tilde{x}) \tilde{\xi}_\alpha(\tilde{x}) \frac{(v_+^n + i v_+^\tau) \omega_+}{v_+^4} \exp\left(-\frac{\nu r}{v_+} e^{-i\phi}\right) \chi(x - \tilde{x}), \end{aligned} \quad (57)$$

where  $\delta/\delta \xi_\alpha(\tilde{x})$  denotes the functional differentiation, and  $\tilde{\omega} = \tilde{u} + i\tilde{w}$ . Together with the linearized evolution equation, Eq. (57) can be used to carry out stability analysis of general steady flame configurations, given by the solutions of Eqs. (30), (56). To the best of our knowledge, no such closed equation as (57) has yet been derived to handle the problem. Its applications will be presented elsewhere.

## VII. DISCUSSION AND CONCLUSIONS

The results presented in this paper solve the problem of non-perturbative *description* of unsteady premixed flame propagation with arbitrary gas expansion. Supplemented by the evolution equation, Eq. (27) gives the closed description of unsteady flames in the most general form. Thus, as in the stationary 2D case, the dilemma mentioned in Introduction is resolved in negative for 2D unsteady flame propagation. The conclusion that the detailed bulk structure of the gas flow is actually unnecessary for describing front dynamics is even more striking in the latter case. Indeed, the highly complicated vortex flow downstream continuously changing in time is naturally expected to have an exceedingly complicated nonlocal influence on the flame front evolution. We proved, however, that all necessary information about this influence is encoded in the complex history of the combination  $M = N \bar{v}_+^n \sigma_+$ . In this connection, a curious circumstance is worth mentioning. As we saw in Sec. III, the vortex component depends on spatial coordinates through the complex combinations  $\tau_\pm$  appearing as the limits of integration in the complex time plane. It is easy to see that for any curved flame configuration, there are always points at the front, where  $\Omega < 0$ . For such points, the time argument of the function  $M(\tilde{x}, t - \tau)$  in the integrand of Eq. (21) has real part  $> t$ . In other words, integration over such points is in a sense looking into the flame future. This does not lead to any conflict with causality, because the corresponding contribution is eventually annihilated by the operator  $(1 + i\hat{\mathcal{H}})$  in Eq. (27). Indeed, the vortical part of  $\mathbf{v}^v$  comes from integration over  $\tilde{x}$  such that the vectors  $\mathbf{r}$  and  $\bar{\mathbf{v}}_+$  are almost parallel, i.e.  $\Omega \approx 1$ , and therefore, the contribution of points with  $\Omega < 0$  is a pure potential satisfying Eq. (23). Retaining this contribution in intermediate formulas is necessary to guarantee continuity of the potential component.

The paradox as to the seemingly “teleological” structure of (27) is closely related to the analyticity properties of  $M(x, t)$  in the complex  $t$ -plane, that allowed us to simplify Eq. (14) to Eq. (18). Our considerations were carried under the very weak assumption of exponential boundedness of this function, which is certainly sufficient for investigation of any flame propagation phenomena. In particular, the linear stability problem analyzed in Sec. V A gives an example of  $M(x, t)$  which is analytic in the complex plane, so that the question whether the contour deformation is legitimate does not arise at all. In this simplest case knowing the exponentials  $\exp(\nu t)$  in Eq. (34) at current time  $t$  allows one to predict their future, so that the “teleological” question does not arise either. Things are more interesting, however, in the case of flame propagation in time-dependent gravitational field. Suppose that the experimentalist plans to leave the burner fall freely at some time instant  $t_0$ . This means that  $g(t)$ , and hence,  $M(\tilde{x}, t)$  [Cf. Eqs. (10), (51)] will have singularities at  $\tau_0 = t_0 \pm i\Delta t$ , where  $\Delta t$  is of the order of duration of the field switching off. As is shown in the Appendix, crossing these singularities by the contour  $C_- \cup C_+$  in Fig. 2 gives rise to a potential contribution that satisfies the conditions a)–c) of Sec. III, and hence does not change Eq. (27), thus resolving the causality issue.

To conclude, Eq. (27) opens a wealth of key developments in theoretical and numerical combustion (extended propagation laws, coupling with acoustics, burners, etc.), not to mention the other fronts evoked at the beginning

of this paper. In particular, Eq. (57) allows direct analytical investigation of small disturbances propagating on steady front patterns such as Bunsen- or V-flames in 2-D configurations. Extension of the above results to the three-dimensional problems is still an open question, one of the main difficulties being the generalization of the  $(1 + i\hat{\mathcal{H}})$  operator projecting out the potential contributions of the burnt-gas flow.

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### APPENDIX A

When deriving the expression (21) for the vortex mode in Sec. III we have omitted the contribution of the contour integral in Eq. (17), retaining only the singular contribution of the poles  $\tau_{\pm}$ . That this operation respects the property c) was already shown in Sec. III A. We will now prove that Eq. (21) does reproduce correctly the near-the-front distribution of vorticity of the burnt gas flow. In essence, the subsequent calculation reproduces the consistency check given in the appendix A of [12]. First of all, taking into account the formula

$$\frac{\partial \tau_{\pm}}{\partial x_i} = \pm \frac{i}{\bar{v}_+} \left( \beta_i \mp i \frac{\bar{v}_{i+}}{\bar{v}_+} \right) \quad (\text{A1})$$

which is readily verified using the definitions (15), (55), one has

$$\begin{aligned} \varepsilon_{ki} \partial_k v_i^v &= \frac{1}{4} \frac{\partial}{\partial x_i} \left\{ \int_{-\infty}^{+\infty} d\tilde{x} \frac{e^{-\mu r}}{\bar{v}_+} \left[ M(\tilde{x}, t - \tau_+) \left( \beta_i - i \frac{\bar{v}_{i+}}{\bar{v}_+} \right) \right. \right. \\ &\quad \left. \left. + M(\tilde{x}, t - \tau_-) \left( \beta_i + i \frac{\bar{v}_{i+}}{\bar{v}_+} \right) \right] \right\}_{\mu=0+} \\ &= \frac{1}{2} \frac{\partial}{\partial x_i} \text{Re} \left\{ \int_{-\infty}^{+\infty} d\tilde{x} \frac{e^{-\mu r}}{\bar{v}_+} M(\tilde{x}, t - \tau_+) \left( \beta_i - i \frac{\bar{v}_{i+}}{\bar{v}_+} \right) \right\}_{\mu=0+}. \end{aligned}$$

Following the argument given in Sec. III, one performs the differentiation with respect to  $x_i$  under the sign of the  $\tilde{x}$ -integral, and sees that the derivative of  $e^{-\mu r}$  leads to an integral proportional to  $\mu$ . Therefore, the only non-zero term contributed by this integral after  $\mu$  is continued to zero is an inessential  $\mathbf{x}$ -independent constant that falls off from the expression  $\text{rot} D\mathbf{v}^v$ . Next, it is easily checked that

$$\left( \beta_k - i \frac{\bar{v}_{k+}}{\bar{v}_+} \right)^2 \equiv 0, \quad (\text{A2})$$

so differentiation of  $M(\tilde{x}, t - \tau_+)$  gives zero, too. Note also that  $\beta$  can be written as

$$\beta_i = \frac{\varepsilon_{ik} \bar{v}_{k+}}{\bar{v}_+} \chi(\varepsilon_{lm} r_l \bar{v}_{m+}), \quad (\text{A3})$$

since it is orthogonal to  $\bar{v}_+$  and changes sign at the point satisfying Eq. (12). Thus, we find

$$\begin{aligned} \varepsilon_{ki} \partial_k v_i^v &= \text{Re} \left\{ \int_{-\infty}^{+\infty} d\tilde{x} \frac{e^{-\mu r}}{\bar{v}_+^2} M(\tilde{x}, t - \tau_+) \varepsilon_{ik} \bar{v}_{k+} \varepsilon_{in} \bar{v}_{n+} \delta(\varepsilon_{lm} r_l \bar{v}_{m+}) \right\}_{\mu=0+} \\ &= \int_{-\infty}^{+\infty} d\tilde{x} M(\tilde{x}, t - r/\bar{v}_+) \delta(\varepsilon_{lm} r_l \bar{v}_{m+}). \end{aligned} \quad (\text{A4})$$

The factor  $e^{-\mu r}$  and the symbol of analytic continuation have been omitted in the last expression because it is explicitly finite. The argument of the  $\delta$ -function turns into zero when the vectors  $r_i$  and  $\bar{v}_{i+}$  are parallel. Near this point, one has approximately

$$\varepsilon_{lm} r_l \bar{v}_{m+} = r \bar{v}_+ \phi.$$

On the other hand, a simple geometric consideration shows that  $d\tilde{x}$  near the same point can be written as (see Fig. 3)

$$d\tilde{x} = -\frac{r \bar{v}_+ d\phi}{N \bar{v}_+^n},$$

where all quantities are taken at the time instant  $t$ . Substituting these expressions into Eq. (A4), and taking into account the relation

$$\delta(\alpha x) = \frac{1}{|\alpha|} \delta(x)$$

yields

$$\varepsilon_{ki} \partial_k v_i^v = \int d\phi \delta(\phi) \frac{M(\tilde{x}, t - r/\bar{v}_+)}{N(\tilde{x}, t) \bar{v}_+^n(\tilde{x}, t)}. \quad (\text{A5})$$

If the observation point is taken at the flame front,  $r$  turns into zero together with  $\phi$ . Recalling also the definition (10) of the function  $M(\tilde{x}, t)$ , we arrive finally at the desired identity

$$(\varepsilon_{ki} \partial_k v_i^v)_+ = \sigma_+(\tilde{x}, t).$$

Let us now return to the question of possibility to perform the contour deformation in the complex  $\tau$ -plane, used in the derivation of Eq. (19). We note, first of all, that this question is concerned entirely with the structure of the potential component of the gas velocity. Indeed, under our general assumption of existence of a short wavelength cutoff, all functions involved are smooth functions of time (because any structure of finite size takes finite time to develop). Hence, these functions (in particular, the function  $M(\tilde{x}, t)$ ) are analytic in a vicinity of the real axis in the complex  $t$ -plane. On the other hand, we know that the value of vorticity in any given point  $(x, y)$  near the front is equal to its value at the point on the front, satisfying  $\Omega = 1$ , in which case the poles  $\tau_{\pm}$  take the real value  $r/\bar{v}_+$ . For  $\tilde{x}$  in a vicinity of that point,  $\tau_{\pm}$  belong to the analyticity domain of  $M(\tilde{x}, t)$ , and hence, of the function  $\mathcal{M}(\tilde{x}, \tau, t)$ . Thus, the use of the Cauchy theorem and the contour deformation performed in Sec. III are legitimate for these  $\tilde{x}$ , while integration over all other  $\tilde{x}$  gives rise to a potential contribution. This, however, does not conclude consideration, because the argument just given proves potentiality of this contribution in some vicinity of the given observation point. To prove it for all  $x \in [0, 1]$ , the contour  $C_- \cup C_+$  in Fig. 2 should be moved to the left of  $\tau_{\pm}$  for all  $\tilde{x}$ . Let us show that this is possible indeed under the assumption used already in the derivation of Eq. (21), namely, that  $M(\tilde{x}, t)$  considered as the function of the complex  $t$  is exponentially bounded near  $t = \infty$ . Suppose, for instance, that this function is meromorphic, i.e., has only poles of arbitrary order in the  $t$ -plane. Any pole of the order  $n$  in the function  $M(\tilde{x}, t)$  becomes an  $(n-1)$ th order pole with respect to  $\tau$  in  $\mathcal{M}(\tilde{x}, \tau, t)$ . Hence, crossing these poles by  $C_- \cup C_+$  does not change the right hand side of Eq. (17) unless  $n = 2$  or  $n = 1$ . Consider first the case  $n = 2$ . Then  $\mathcal{M}(\tilde{x}, \tau, t)$  has simple poles at some point  $\tau_0$  and its complex conjugate  $\tau_0^*$  ( $\tau_0$  may depend on  $\tilde{x}, t$ , but we do not write this dependence explicitly, for brevity). We are to show that their contribution to the right hand side of Eq. (17), given by the Cauchy theorem as (“c.c.” stands for complex conjugate)

$$\frac{2\pi i}{4\pi} \text{res } \mathcal{M}(\tilde{x}, \tau_0, t) \left\{ \frac{1}{\tau_0 - \tau_+} + \frac{1}{\tau_0 - \tau_-} \right\} + \text{c.c.}, \quad (\text{A6})$$

gives rise to a field  $V_i$  that satisfies conditions a) - c). Substituting this expression into Eq. (11) yields (we do not introduce intermediate regularization because the  $\tilde{x}$ -integral will be shown to converge)

$$V_i = -\frac{i}{4} \varepsilon_{ik} \int_{-A}^{+A} d\tilde{x} \text{res } \mathcal{M}(\tilde{x}, \tau_0, t) \partial_k \left\{ \frac{1}{\tau_0 - \tau_+} + \frac{1}{\tau_0 - \tau_-} \right\} + \text{c.c.} \quad (\text{A7})$$

The property a) is evidently satisfied. To prove b) we write, using Eqs. (A1), (A2) and (A3)

$$\begin{aligned}
& \partial_k \int_{-A}^{+A} d\tilde{x} \operatorname{res} \mathcal{M}(\tilde{x}, \tau_0, t) \partial_k \left\{ \frac{1}{\tau_0 - \tau_+} + \frac{1}{\tau_0 - \tau_-} \right\} \\
&= \int_{-A}^{+A} d\tilde{x} \operatorname{res} \mathcal{M}(\tilde{x}, \tau_0, t) \left\{ \frac{\partial_k^2 \tau_+}{(\tau_0 - \tau_+)^2} + \frac{\partial_k^2 \tau_-}{(\tau_0 - \tau_-)^2} \right\} \\
&= 2i \int_{-A}^{+A} d\tilde{x} \operatorname{res} \mathcal{M}(\tilde{x}, \tau_0, t) \delta(\varepsilon_{lm} r_l \bar{v}_{m+}) \left\{ \frac{1}{(\tau_0 - \tau_+)^2} - \frac{1}{(\tau_0 - \tau_-)^2} \right\} = 0,
\end{aligned}$$

since  $\tau_+ = \tau_-$  when the argument of the delta function is zero. Thus,  $\operatorname{rot} \mathbf{V} = 0$ . Last,  $\mathcal{M}(\tilde{x}, \tau, t)$  is periodic in  $\tilde{x}$ , and therefore, so is its pole. Taking into account also that  $\tau_{\pm} = O(|\tilde{x}|)$ ,  $\partial_i \tau_{\pm} = O(1)$  for  $|\tilde{x}| \rightarrow \infty$ , one sees that the  $\tilde{x}$ -integral in Eq. (A7) is convergent in the limit  $A \rightarrow \infty$  for all  $\mathbf{x}$ .

In the case  $n = 1$  the function  $\mathcal{M}(\tilde{x}, \tau, t)$  contains a logarithmic singularity of the form  $a(\tilde{x}, t) \ln(\tau - \tau_0)$  (and its complex conjugate). Crossing this singularity leads to the  $2\pi$  jump in  $\arg(\ln(\cdot))$  for all points of the contour  $C_- \cup C_+$ , located at one side of the point  $\tau_0$ . Hence, expression (A6) is replaced in this case by the following

$$\frac{2\pi i a(\tilde{x}, t)}{4\pi} \int_0^{\tau_0} d\tau \left\{ \frac{1}{\tau - \tau_+} + \frac{1}{\tau - \tau_-} \right\} + \text{c.c.}$$

The proof of the properties a)–c) is exactly the same as before.

Let us finally consider the case when the function  $M(\tilde{x}, t)$  has branch singularities. If these singularities are connected by a number of cuts so that  $M(\tilde{x}, t)$  is meromorphic in the cut  $\tau$ -plane, then so is the function  $\mathcal{M}(\tilde{x}, \tau, t)$ , and moving the contour of integration beyond a cut results in a contribution to the right hand side of Eq. (17) of the form

$$\frac{1}{4\pi} \int_{C_0} d\tau [\mathcal{M}](\tilde{x}, \tau, t) \left\{ \frac{1}{\tau - \tau_+} + \frac{1}{\tau - \tau_-} \right\} + \text{c.c.},$$

where  $[\mathcal{M}](\tilde{x}, \tau, t)$  denotes the jump<sup>5</sup> of the function  $\mathcal{M}(\tilde{x}, \tau, t)$  across the cut  $C_0$ . If this cut has finite length, then the above considerations again apply literally. However, in the case of an infinite cut, the  $\tau$ -integral is apparently divergent. This means that such cuts, if any, are only allowed in regions where  $\mathcal{M}(\tilde{x}, \tau, t)$  satisfies more restrictive conditions than the exponential boundedness. We do not pursue details here, because physical significance of such cuts is not clear.

Thus, the function  $M(\tilde{x}, t)$  is allowed to have any number of branch singularities in the complex  $t$ -plane, connected by cuts of finite length, as well as any number of poles of arbitrary order to justify the contour deformation used in Sec. III.

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- [1] L. D. Landau and E. Lifschitz, *Fluid Mechanics* (Pergamon Press, New York, 1987).
  - [2] P. Clavin, L. Masse, and F. A. Williams, *Comb. Sci. Techn.* **177**, 979 (2005).
  - [3] W. Hillebrandt and J. C. Niemeyer, *Annu. Rev. Astron. Astrophys.* **38**, 191 (2000).
  - [4] E. Mallard and H. L. Le Chatelier, *C. R. Acad. Sci. (Paris)* **93**, 145 (1881).
  - [5] G. Darrieus (1938), unpublished work presented at *La Technique Moderne*, Paris.
  - [6] L. D. Landau, *Acta Physicochimica USSR* **19**, 77 (1944).
  - [7] G. I. Sivashinsky, *Acta. Astron.* **4**, 1177 (1977).
  - [8] K. A. Kazakov and M. A. Liberman, *Phys. Fluids* **14**, 1166 (2002).
  - [9] M. L. Frankel, *Phys. Fluids* **A2**, 1879 (1990).

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<sup>5</sup> If  $\mathcal{M}(\tilde{x}, \tau, t)$  is singular at  $C_0$ , the above integral can be replaced by the integral of this function over a closed contour embracing the cut  $C_0$ .



- [10] G. H. Markstein, J. Aeron. Sci. **18**, 199 (1951).
- [11] K. A. Kazakov, Phys. Rev. Lett. **94**, 094501 (2005).
- [12] K. A. Kazakov, Phys. Fluids **17**, 032107 (2005).
- [13] G. H. Markstein, Nonsteady flame propagation (Pergamon, New York, 1964).
- [14] G. I. Sivashinsky and P. Clavin, J. Phys. (Paris) **48**, 193 (1987).
- [15] Y. B. Zel'dovich, A. G. Istratov, N. I. Kidin, and V. B. Librovich, Combust. Sci. Technol. **24**, 1 (1980).
- [16] G. Searby and D. Rochwerger, J. Fluid Mech. **231**, 529 (1991).
- [17] V. V. Bychkov and M. A. Liberman, Phys. Rep. **325**, 115 (2000).
- [18] M. Matalon and B. J. Matkowsky, J. Fluid Mech. **124**, 239 (1982).
- [19] P. Pelce and P. Clavin, J. Fluid Mech. **124**, 219 (1982).
- [20] G. Joulin and P. Cambray, Combust. Sci. Tech. **81**, 243 (1992).
- [21] W. D. Hayes, J. Fluid Mech. **2**, 595 (1957).

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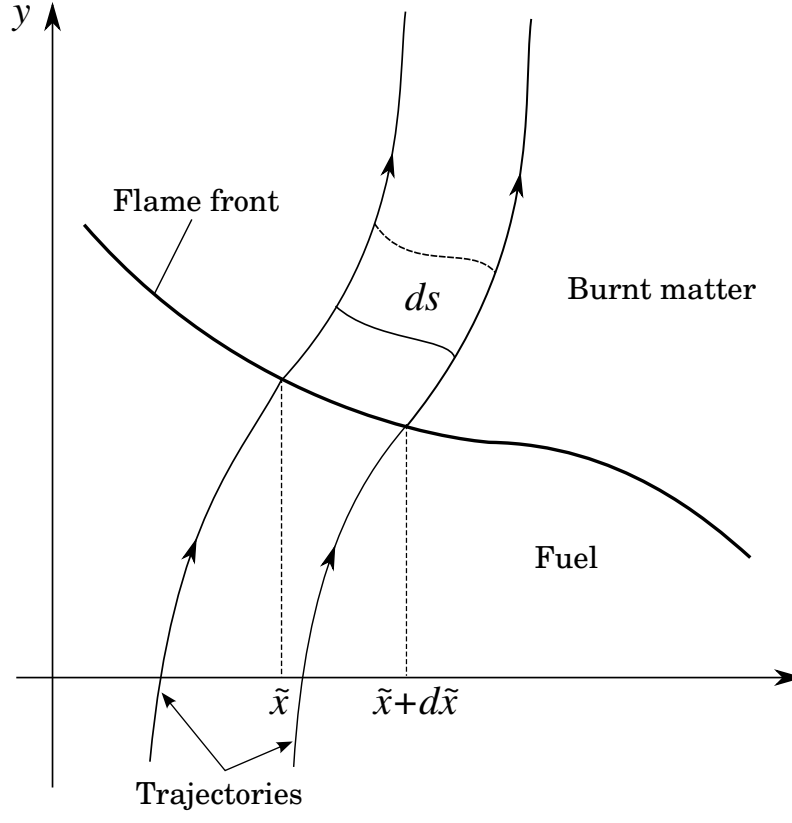


FIG. 1: Elementary decomposition of the flow downstream used in the derivation of the expression (7).

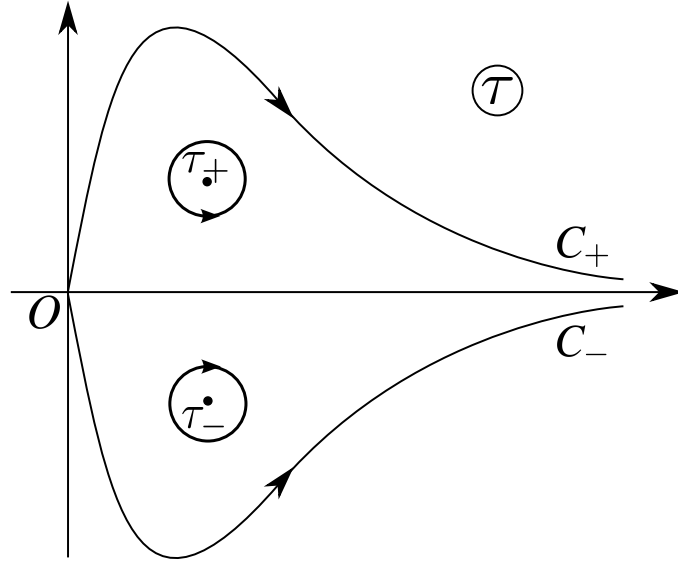


FIG. 2: Extraction of the singularity in Eq. (14) by contour deformation in the complex  $\tau$ -plane.

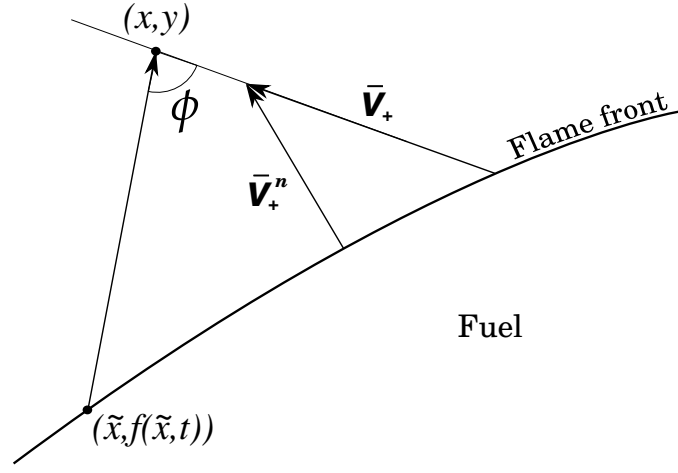


FIG. 3: Near-the-front structure of the flow downstream.